Partial Solution Set, Leon §6.6

6.6.1 Find the matrix associated with each of the quadratic forms.

(a) $3x^2 - 5xy + y^2$

(b) $2x^2 + 3y^2 + z^2 + xy - 2xz + 3yz$

(c) $x^2 + 2y^2 + z^2 + xy - 2xz + 3yz$

Solution:

(a) $A = \begin{bmatrix} 3 & -5/2 \\ -5/2 & 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 1/2 & -1 \\ 1/2 & 3 & 3/2 \\ -1 & 3/2 & 1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 1 & 1/2 & -1 \\ 1/2 & 2 & 3/2 \\ -1 & 3/2 & 1 \end{bmatrix}$.

6.6.6 Which of the following matrices are positive definite? Negative definite? Indefinite?

(a) $A = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$ is positive definite. Since it is 2×2, any test will do.

(b) $A = \begin{bmatrix} 3 & 4 \\ 4 & 1 \end{bmatrix}$ is indefinite. Its determinant is -13, so it has eigenvalues of both signs.

(d) $A = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ is negative definite, since $-A$ is positive definite.

6.6.7 For each of the following functions, determine whether the given stationary point corresponds to a local minimum, a local maximum, or a saddle point.

(d) $f(x, y) = \frac{y}{x^2} + \frac{x}{y^2} + xy$, $(1, 1)$.

(e) $f(x, y, z) = x^3 + xyz + y^2 - 3x$, $(1, 0, 0)$.

Solution:

(d) The Hessian of f at the given stationary point is

$$H = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix},$$

with eigenvalues 3 and 9; both are positive, so the stationary point corresponds to a minimum.

(e) The Hessian of f at the given stationary point is

$$H = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

with eigenvalues $6, 1 \pm \sqrt{2}$; two are positive, but one is negative, so the stationary point corresponds to a saddle point.

6.6.8 Show that if A is symmetric positive definite, then $|A| > 0$. Give an example of a 2×2 matrix with positive determinant that is not positive definite.

Solution: If A is $n \times n$ symmetric positive definite, with eigenvalues $\lambda_1, \ldots, \lambda_n$, then we know that $\lambda_i > 0$ for each $1 \le i \le n$. But then $|A| = \prod_{i=1}^{n} \lambda_i > 0$. □

To show that the converse is false, consider $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

6.6.10 Let A be a *singular* $n \times n$ matrix. Show that $A^T A$ is positive semidefinite, but not positive definite.

Proof: Since the rank of A is smaller than n, it follows that 0 is an eigenvalue of A, and necessarily (check this) 0 is an eigenvalue of $A^T A$. So it is not possible that $A^T A$ is positive definite. Now $A^T A$ is symmetric (Hermitian and normal), so the (suitably chosen) eigenvectors of A constitute an orthonormal basis for R^n. It follows that for any $\mathbf{x} \in R^n$ we can express $\mathbf{x}$ uniquely as a linear combination of the eigenvectors of A. It suffices to show that if $\mathbf{x}$ is an eigenvector of A, with associated eigenvalue λ, then $\mathbf{x}^T A^T A \mathbf{x} \ge 0$. But if $\mathbf{x}$ is an eigenvector of A with associated eigenvalue λ, then

$$\begin{aligned} \mathbf{x}^T A^T A \mathbf{x} &= (A\mathbf{x})^T A\mathbf{x} \\ &= (\lambda\mathbf{x})^T \lambda\mathbf{x} \\ &= \lambda^2 \mathbf{x}^T \mathbf{x} \\ &= \lambda^2 \|\mathbf{x}\|^2 \\ &\ge 0, \end{aligned}$$

and the proof is complete. □